ASYMMETRIC RADIATIVE-CONVECTIVE HEATING OF AN INFINITE PLATE

A. L. Burka

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 126–127, 1966

Problems involving the solution of the heat-conduction equation with nonlinear boundary conditions have recently become of practical, as well as theoretical, interest. This is due to the fact that in many modern applications one must take into account radiative boundary conditions. Despite the difficulties associated with the solution of nonlinear functional equations, the reduction of the boundary value problem to such equations makes it possible to gain additional information about the characteristic variables of the problem.

In addition to the temperature field, one such variable of considerable importance is the heat flux. In this paper we consider the asymmetric heating of a plane-parallel infinite plate of thickness R.

The boundary value problem considered here can be reduced to a system of two nonlinear integral equations of Volterra type for the net radiative heat flux. The problem can be formulated as follows:

$$\frac{\partial T}{\partial \tau} = a\frac{\partial^2 T}{\partial x^2} \qquad (0 \leqslant x \leqslant R,\ 0 \leqslant \tau < \infty)$$

$$\left(\frac{\partial T}{\partial x}\right)_{x=0} = E_0(0,\tau), \qquad \left(\frac{\partial T}{\partial x}\right)_{x=R} = E_1(R,\tau), \qquad T(x,0) = T_0,$$

$$E_0(0,\tau) = H\,[T^4(0,\tau) - T_1^4],$$

$$E_1(R,\tau) = h\,[T_2 - T(R,\tau)] - HT^4(R,\tau),$$

$$(T_0,\ T_1,\ T_2 = \text{const}). \tag{1}$$

Here H is a reduced radiative heat transfer coefficient, a is the thermal diffusivity, T_0 is the initial temperature of the plate, h is a reduced convective heat transfer coefficient, T_1 is the absolute temperature of the radiative source, and T_2 is the absolute temperature of the surroundings.

Applying the Laplace transform to (1), we obtain the general solution of the problem in the transform plane

$$u(x,p) = \frac{T_0}{p} + v_1(R,p)\frac{\operatorname{ch}(\sqrt{p/a}\,x)}{\sqrt{p/a}\operatorname{sh}(\sqrt{p/a}\,R)} -$$

$$- v_0(0,p)\frac{\operatorname{ch}[\sqrt{p/a}\,(x-R)]}{\sqrt{p/a}\ \operatorname{sh}\sqrt{p/a}\,R},$$

$$u(x,p) = \int_0^\infty T(x,\tau)\exp(-p\tau)\,d\tau,$$

$$v_0(0,p) = \int_0^\infty E_0(0,\tau)\exp(-p\tau)\,d\tau,$$

$$v_1(R,p) = \int_0^\infty E_1(R,\tau)\exp(-p\tau)\,d\tau\ . \tag{2}$$

Applying the inverse transformation to (2), we obtain an expression for the temperature field in the plate, which in dimensionless form reads

$$\theta(x,\tau) = \tag{3}$$

$$1 + \alpha_3\int_0^\tau\left\{1 + 2\sum_{n=1}^\infty(-1)^n\cos\frac{\pi n}{R}x\exp[-\alpha_4 n^2(\tau - t)]\right\}\varphi_1(R,t)\,dt -$$

$$-\alpha_5\int_0^\tau\left\{1 + 2\sum_{n=1}^\infty(-1)^n\cos\frac{\pi n}{R}(R - x)\exp[-\alpha_4 n^2(\tau - t)\right\}\varphi_0(0,t)\,dt\,.$$

Equation (3) is a formal solution for the temperature field $\theta(x,\tau)$ in terms of the net radiative heat fluxes $\varphi_0(0,\tau)$, $\varphi_1(R,\tau)$. Thus in order to calculate the temperature field one must first determine the radiative heat fluxes. From (3) we obtain a system of two nonlinear integral equations for the net radiative fluxes φ_0 and φ_1.

$$\varphi_0(0,\tau) =$$

$$\alpha_2\left\{1 + \alpha_3\int_0^\tau\left\{1 + 2\sum_{n=1}^\infty(-1)^n\exp[-\alpha_4 n^2(\tau - t)]\right\}\varphi_1(R,t)\,dt -\right.$$

$$\left.= \alpha_5\int_0^\tau\left\{1 + 2\sum_{n=1}^\infty\exp[-\alpha_4 n^2(\tau - t)]\right\}\varphi_0(0,t)\right\}^4 dt - \alpha_1,$$

$$\varphi_1(R,\tau) =$$

$$= \alpha_8 - \alpha_6\left\{1 + \alpha_3\int_0^\tau\left\{1 + 2\sum_{n=1}^\infty\exp[-\alpha_4 n^2(\tau - t)]\right\}\varphi_1(R,t)\,dt -\right.$$

$$\left.-\alpha_5\int_0^\tau\left\{1 + 2\sum_{n=1}^\infty(-1)^n\exp[-\alpha_4 n^2(\tau - t)]\right\}\varphi_0(0,t)\right\}dt -$$

$$-\alpha_7\left\{1 + \alpha_3\int_0^\tau\left\{1 + 2\sum_{n=1}^\infty\exp[-\alpha_4 n^2(\tau - t)]\right\}\varphi_1(R,t)\,dt -\right.$$

$$\left.-\alpha_5\int_0^\tau\left\{1 + 2\sum_{n=1}^\infty(-1)^n\exp[-\alpha_4 n^2(\tau - t)]\right\}\varphi_0(0,t)\right\}^4 dt,$$

$$\alpha_1 = \frac{T_1^4}{T_1^4 + T_0^4}, \qquad \alpha_2 = \frac{T_0^4}{T_1^4 - T_0^4}, \qquad \alpha_3 = \frac{a}{RT_0}[h(T_2 - T_0) - HT_0^4],$$

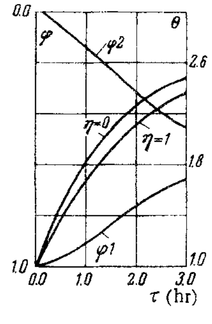
$$\alpha_4 = \frac{\pi^2 a}{R^2}, \qquad \alpha_5 = \frac{a}{RT_0}H(T_1^4 - T_0^4), \qquad \alpha_6 = \frac{hT_0}{h(T_2 - T_0) - HT_0^4},$$

$$\alpha_7 = \alpha_6\frac{HT_0^4}{\mu T_0}, \quad \alpha_8 = \alpha_6\frac{T_2}{T_0},$$

$$\theta(x,\tau) = \frac{T(x,\tau)}{T_0}, \qquad \varphi_0(0,\tau) = \frac{E_0(0,\tau)}{H(T_1^4 - T_0^4)},$$

$$\varphi_1(R,\tau) = \frac{E_1(R,\tau)}{h(T_2 - T_0) - HT_0^4}\,. \tag{4}$$

To solve this problem, we reduce (3) and (4) to finite-difference form. We divide the interval $[0, \tau]$ into m equal parts and assume φ_0 and φ_1 to be constant over each subinterval. In other words, for the sake of simplicity we use the rectangular-step formula, although in principle we could use any quadrature formula of higher accuracy.



The accuracy of our approximation depends on the size of the step $\Delta\tau$, i.e.,

$$\lim_{m\to\infty}\varphi_0(0,\Delta\tau m) \to \varphi_0(0,\tau), \quad \lim_{m\to\infty}\varphi_1(R,\Delta\tau m) \to \varphi_1(R,\tau).$$

As is well known, one of the most effective methods of solution of nonlinear functional equations is Newton's method. This method has been extended to the solution of systems of nonlinear algebraic equations. The conditions for the convergence of this method have been fully discussed by Kantorovich [1]. Assuming that the finite-difference form of (4) satisfies all the conditions necessary for the convergence of the iteration, we find the roots $\varphi_0(0, \Delta\tau m)$ and $\varphi_1(R, \Delta\tau m)$ for every m by Newton's method. Thus, using the finite-difference form of (3) and (4), we obtain the values of the net radiative fluxes and, consequently, the temperature field for arbitrary $\tau = \Delta\tau m$.

The figure shows $\varphi_0(0, \tau)$, $\varphi_1(R, \tau)$, $\theta(\eta, \tau)$ as functions of time for the following geometrical and thermophysical parameters:

$$\eta = x / R, \quad R = 0.03 \text{ m},$$

$$a = 3.3 \cdot 10^{-3} \text{ m}^2/\text{hr}, \quad T_0 = 253^\circ \text{ K},$$

$$T_1 = 815^\circ \text{ K}, \quad T_2 = 300^\circ \text{ K},$$

$$h = 1 \text{ m}^{-1}, \quad H = 4.188 \cdot 10^{-9} \text{ deg C}^3/\text{m}.$$

The computations were carried out on the M-20 computer of the Computing Center of the Siberian Branch, Academy of Sciences USSR. The error of the results does not exceed 10^{-5}.

REFERENCE

1. L. V. Kantorovich, "On Newton's method," Tr. Matem. in-ta AN SSSR, vol. 28, 1949.